THETA SERIES AND TRACE OPERATORS FOR $\Gamma_n[q]$

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Abstract. We consider the action of suitable trace operators on non homogeneous theta series that are Siegel modular forms for the principal congruence subgroups of the symplectic group of level q, $\Gamma_n[q]$. This is used for investigating whether modular forms for $\Gamma_n[N]$, with $N \mid q$, that are linear combination of such theta series, can be expressed as combination of theta series that are modular forms with respect to $\Gamma_n[N]$.

1. Introduction

Aim of this work is to carry on the study of the action of trace operators on theta series. In particular, we are going to deal with the case of theta series relevant to the principal congruence subgroups $\Gamma_n[q]$. Previously trace operators have been studied with respect to couples of Hecke subgroups (of varying level) and couples formed by Hecke and principal congruence subgroups of the same level (see [**B.F.S.-P.**], [**Ku.**] and [**Ch.**] resp.).

Let us now briefly recall the context of our investigations. First, we fix the notation: if Γ is a congruence subgroup of $Sp(n,\mathbb{Z})$ and if we denote by $[\Gamma,\rho,\chi]$ the space of modular forms for Γ with respect to the rational represntation $\rho=[\rho_0,r]$, and multiplier system χ, we shall denote $\Theta[\Gamma]_\rho \subset [\Gamma,\rho,\chi]$ the subspace spanned by the theta series that is convenient to consider in relation with Γ. It is in fact well known that theta series attached to quadratic forms with rational coefficients, positive definite, and if necessary endowed with some characteristics, provide families of (Siegel) modular forms relative to

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the congruence subgroups of $Sp(n,\mathbb{Z})$ belonging to all the most important infinite classes; for example, this is true for the Hecke subgroups $\Gamma_{n,0}[q]$, for the principal congruence subgroups $\Gamma_n[q]$, and for the Igusa subgroups $\Gamma_n[q,2q]$.

If one considers a pair of congruence subgroups $\Gamma' \supset \Gamma$ and a pair of multiplier systems χ' on Γ' and χ on Γ such that $\chi'|_\Gamma = \chi$, it is quite natural to ask if the following relation holds:

$$\Theta[\Gamma]_\rho \cap [\Gamma', \rho, \chi'] = \Theta[\Gamma']_\rho \ . \tag{1}$$

A particular motivation for the study of such a question is given by some results in adelic representation theory which imply the represantability of modular forms as linear combination of theta series of not fixed type (see [**S.P.**]). It is then desirable to understand whether it is possible to turn linear combinations of "generic" theta series into combinations of the convenient theta series.

We can translate the question (1) into different terms by introducing a suitable symmetrization map which is in fact the trace operator. We shall actually investigate if it is possible to express the image of a theta series in $\Theta[\Gamma]_\rho$ under the action of the trace operator as a linear combination of theta series in $\Theta[\Gamma']_\rho$.

In this work we are going to examine the case of pairs of the kind: $\Gamma_n[N] \supset \Gamma_n[q]$.

The key step will be the computation of a commutation formula between the trace operator and the Siegel Φ-operator. This will allow us to transfer the problem from the given degree to the singular case and, hence, we shall just have to use known results on singular modular forms. Important tools to obtain such commutation formulas will be a careful description of the quotient group $\frac{\Gamma_n[N]}{\Gamma_n[q]}$ and an inductive argument on the rank of the characteristics. Basically this kind of strategy was used for the first time in the paper [**S.M.**] where it was introduced in relation with the study of rings of stable modular forms and of modular forms obtained as products of Thetanullwerte.

2. Background and definitions

Let us start with the introduction of the trace operator. If $\Gamma \leq \Gamma'$ are two congruence subgroups of $Sp(n,\mathbb{Z})$, $2k = r \in \mathbb{Z}$, ρ_0 is a rational reduced representation of $GL(n,\mathbb{C})$, $\rho = \rho_0 \otimes \det^r$, or rather with the same notation as in [**Fr.1**] $\rho = [\rho_0, r]$, and, finally, v and v' are two multiplier systems for Γ and Γ', respectively, such that $v'|_\Gamma = v$, then it is clear that $[\Gamma, \rho, v] \supseteq [\Gamma', \rho, v']$.

2.1. Definition *With the notation just introduced, given $f \in [\Gamma, \rho, v]$ we can define the trace of f :*

$$Tr^{\Gamma,v}_{\Gamma',v'} f = \frac{1}{[\Gamma' : \Gamma]} \sum_{g \in \Gamma \backslash \Gamma'} v'(g)^{-1} f|_\rho g. \tag{2}$$

Where, as usual, for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{Z})$,

$$f|_\rho g(Z) := \rho(CZ + D)^{-1} f(g\langle Z \rangle).$$

During the explicit calculations we are going to carry out, the normalization factor $\frac{1}{[\Gamma':\Gamma]}$ will be omitted.
It is obvious that the trace turns out to be a surjective linear map from $[\Gamma, \rho, v]$ onto $[\Gamma', \rho, v']$.

Now we illustrate in which way the trace operator can be useful to understand if it is possible to express a given modular form as a linear combination of suitable theta series. Let $f \in \Theta[\Gamma]_k \cap [\Gamma', \rho, v']$, and let $\{\vartheta_i\}_{i \in I}$ be a set of series in $\Theta[\Gamma]_k$ such that f is a linear combination of these ϑ_i's. If for every ϑ_i it happens that $Tr^{\Gamma,v}_{\Gamma',v'} \vartheta_i$ can be expressed as a linear combination of theta series $\theta_j \in [\Gamma', \rho, v']$, then, clearly, for $f = Tr^{\Gamma,v}_{\Gamma',v'} f$ too, there will be such an expression.

In the sequel we shall only deal with the following case:

$$\Gamma = \Gamma_n[q] \leq \Gamma_n[N] = \Gamma',\ q = N \cdot p,\ p \text{ prime},$$

where we recall that $\Gamma_n[q]$ is the principal congruence subgroup and is defined as

$$\Gamma_n[q] = \{M \in Sp(n, \mathbb{Z}) \mid M \equiv E \bmod q\}.$$

(E denotes the $(2n \times 2n)$ identity matrix.)

To fix our notations, we give the general expression of the theta series we are going to consider. Given a harmonic form P with respect to ρ_0, a positive definite, rational quadratic form $S = S^{(r,r)}$ and some rational characteristics $U,\ V \in \mathbb{Q}^{(r,n)}$, we will denote by $\vartheta^{(n)}_P \begin{bmatrix} U \\ V \end{bmatrix} (S, Z)$ the function defined as follows:

$$\sum_{G \in \mathbb{Z}^{(r,n)}} P(S^{\frac{1}{2}}(G + U)) \exp(\pi i \sigma(S[G + U]Z + 2\,{}^tV(G + U))).$$

(With the standard notation $S[G] = {}^tGSG$.) Next, we want to specify which theta series will play a major part in relation to the principal congruence subgroups. As to the level q, it is convenient take into account theta series of the following type:

$$\vartheta_P^{(n)}\left[\begin{array}{c}0\\ \frac{V}{q}\end{array}\right](S/q,Z):=\sum_{G\in\mathbb{Z}^{(r,n)}}P((S/q)^{\frac{1}{2}}G)\exp(\frac{\pi i}{q}\sigma(S\,[G]\,Z+2\,{}^tVG))$$

where $S=S^{(r,r)}$ is an even positive definite quadratic form such that q^2S^{-1} is even and the characteristic $V\in\mathbb{Z}^{(r.n)}$ satisfies the relations i) $S^{-1}\,[V]$ is even; ii) $qS^{-1}V$ is integral. It follows from the general theta transformation formalism that $\vartheta_P^{(n)}\left[\begin{array}{c}0\\ \frac{V}{q}\end{array}\right](S/q,Z)\ \in\ [\Gamma_n\,[q]\,,\rho,v_{S/q}]$; the reason for considering such theta series is given from the fact that, with some restrictions on the multiplier system, they span the spaces of singular modular forms with respect to $\Gamma_n\,[q]$ (see [**Fr.1**] for details).

For our purposes another kind of theta series is also useful.

2.2. Definition *Let Q be an integral positive definite quadratic form, and q an integer such that qQ^{-1} is even. We then put*

$$T_{n,q}(Q):=\left\{T\in\mathbb{Z}^{(r,n)}\mid QT\equiv 0\bmod q\right\}/\bmod q.$$

Then we consider:

$$\vartheta_P\,(Z,Q\,|\,T):=\vartheta_P\left[\begin{array}{c}\frac{T}{q}\\ 0\end{array}\right](Q,Z)\quad (T\in T_{n,q}(Q)).$$

2.3. Remark As we have seen before, if S and q^2S^{-1} are even and $V=V^{(r,n)}$ is an integral matrix such that $S^{-1}\,[V]$ is even and $qS^{-1}V$ is integral, then the associated theta series $\vartheta_P\left[\begin{array}{c}0\\ \frac{V}{q}\end{array}\right](S/q,Z)\in\left[\Gamma_n\,[q]\,,\rho,v_{S/q}\right]$. Next, we consider an integral matrix $A\in GL(r,\mathbb{Q})$ such that $S\mathbb{Z}^{(r)}+V\mathbb{Z}^{(n)}+q\mathbb{Z}^{(r)}=A\mathbb{Z}^{(r)}$ (as subgroups of $\mathbb{Z}^{(r)}$) and the positive definite quadratic form $\tilde{S}=S^{-1}[A]=S[S^{-1}A]$. It is easily seen that qA^{-1} and $A^{-1}S$ are integral whereas $\tilde{S}$ is even. Moreover, since $\tilde{S}^{-1}=S[\,{}^tA^{-1}]=S^{-1}[S\ {}^tA^{-1}]$, $q\tilde{S}^{-1}$ is integral and $q^2\tilde{S}^{-1}$ is even. In particular, if q is odd then $q\tilde{S}^{-1}$ is even. We can now apply the Mumford Identity (see [**Mu.**])to the pair of symmetric matrices S, $\tilde{S}$, obtaining

$$\vartheta_P^{(n)}\left[\begin{array}{c}0\\ \frac{V}{q}\end{array}\right](S/q,Z)=\sum_{T\in{}^tAS^{-1}\mathbb{Z}^{(r)}/\mathbb{Z}^{(r)}}\vartheta_P^{(n)}\left[\begin{array}{c}0\\ \frac{{}^tAS^{-1}V+qT}{q}\end{array}\right]\left(\tilde{S}/q,Z\right).$$

Also, if we notice that ${}^tAS^{-1}=\tilde{S}A^{-1}$, we get from this equality that $\vartheta_P^{(n)}\left[\begin{array}{c}0\\ \frac{V}{q}\end{array}\right](S/q,Z)$ is a combination (in fact a sum) of theta series of type

$\vartheta_P^{(n)}\left[\begin{array}{c}0\\ \frac{\tilde{S}Y}{q}\end{array}\right]\left(\tilde{S}/q,Z\right)$ $(Y=A^{-1}(V+qX),\,Y,\,X\in\mathbb{Z}^{(r,n)})$.

We now want to express in a different manner these last theta series.

$$\vartheta_P^{(n)}\left[\begin{array}{c}0\\ \frac{\tilde{S}Y}{q}\end{array}\right]\left(\tilde{S}/q,Z\right)=$$

$$=\sum_{G\in\mathbb{Z}^{(r,n)}}P((\tilde{S}/q)^{\frac{1}{2}}G)\exp(\tfrac{\pi i}{q}\sigma(\,\tilde{S}\,[G]\,Z+2\,{}^t(\tilde{S}Y)G))=$$

$$=\sum_{\alpha\in\mathbb{Z}^{(r,n)}}\ \sum_{\beta\in\mathbb{Z}^{(r,n)}/q\tilde{S}^{-1}\mathbb{Z}^{(r,n)}}P((\tilde{S}/q)^{\frac{1}{2}}(q\tilde{S}^{-1}\alpha+\beta))\cdot$$

$$\cdot\exp(\tfrac{\pi i}{q}\sigma(\,\tilde{S}\left[q\tilde{S}^{-1}\alpha+\beta\right]Z++2\,{}^t(\tilde{S}Y)\left(q\tilde{S}^{-1}\alpha+\beta\right)))=$$

$$=\sum_{\alpha\in\mathbb{Z}^{(r,n)}}\ \sum_{\beta\in\mathbb{Z}^{(r,n)}/q\tilde{S}^{-1}\mathbb{Z}^{(r,n)}}P((q\tilde{S}^{-1})^{\frac{1}{2}}(\alpha+(\tilde{S}/q)\beta))\cdot$$

$$\cdot\exp(\pi i\sigma(q\,\tilde{S}^{-1}\left[\alpha+(\tilde{S}/q)\beta\right]Z+2\,{}^tY\tilde{S}\beta/q))=$$

$$=\sum_{\beta\in\mathbb{Z}^{(r,n)}/q\tilde{S}^{-1}\mathbb{Z}^{(r,n)}}\exp(\pi i\sigma(2\,{}^tY\tilde{S}\beta/q))\vartheta_P^{(n)}\left[\begin{array}{c}\frac{\tilde{S}\beta}{q}\\ 0\end{array}\right]\left(q\tilde{S}^{-1},Z\right).$$

According to Definition 2.2, we can use for the above sum the different notation:

$$\cdots=\sum_{\beta\in\mathbb{Z}^{(r,n)}/q\tilde{S}^{-1}\mathbb{Z}^{(r,n)}}\exp(\pi i\sigma(2\,{}^tY\tilde{S}\beta/q))\vartheta_P^{(n)}\left(Z,q\tilde{S}^{-1}\,|\,\tilde{S}\beta\right).$$

On the other hand, given Q even, or just integral if q is even, such that qQ^{-1} is even and $T\in T_{n,q}(Q)$ i.e., $T=qQ^{-1}L,\ L\in\mathbb{Z}^{(r,n)}$), and the associated $\vartheta_P\,(Z,Q\,|\,T)$ we have:

$$\vartheta_P\,(Z,Q\,|\,T)=$$

$$=\sum_{G\in\mathbb{Z}^{(r,n)}}P(Q^{\frac{1}{2}}(G+Q^{-1}L))\exp(\pi i\sigma(Q\,[G+Q^{-1}L]\,Z))=$$

$$=\sum_{G\in\mathbb{Z}^{(r,n)}}P(Q^{-\frac{1}{2}}(QG+L))\exp(\pi i\sigma(Q^{-1}\,[QG+L]\,Z))=$$

$$=\sum_{G\equiv L\,\mathrm{mod}\,Q}P(Q^{-\frac{1}{2}}G)\exp(\pi i\sigma(Q^{-1}\,[G]\,Z))=$$

$$=K(Q,n)\sum_{G\in\mathbb{Z}^{(r,n)}}\ \sum_{\Lambda\,\mathrm{mod}\,Q}P(Q^{-\frac{1}{2}}G)\exp(\pi i\sigma(Q^{-1}\,[G]\,Z+$$

$$+2^t\Lambda Q^{-1}(G-L)))=$$

$$=K(Q,n)\sum_{\Lambda\,\mathrm{mod}\,Q}\exp(-\tfrac{2\pi i}{q}\sigma({}^t\Lambda T))\vartheta_P^{(n)}\left[\begin{array}{c}0\\ \frac{qQ^{-1}\Lambda}{q}\end{array}\right]((qQ^{-1})/q,Z)\,.$$

We remark that qQ^{-1} and $q^2(qQ^{-1})^{-1} = qQ$ are even, $q(qQ^{-1})^{-1}qQ^{-1}\Lambda = q\Lambda$ is integral and $(qQ^{-1})^{-1}[qQ^{-1}\Lambda] = qQ^{-1}[\Lambda]$ is even, hence we can conclude:

$$\vartheta_P^{(n)} \left[\begin{array}{c} 0 \\ \frac{qQ^{-1}\Lambda}{q} \end{array} \right] \big((qQ^{-1})/q, Z\big) \in [\Gamma_n[q], \rho, v_{Q^{-1}}].$$

As a consequence we get the following result.

2.4. Proposition *The vector space spanned by the theta series* $\vartheta_P^{(n)} \left[\begin{array}{c} 0 \\ \frac{V}{q} \end{array} \right] (S/q, Z)$ *(with S and q^2S^{-1} even and $V = V^{(r,n)}$ integral such that $S^{-1}[V]$ is even and $qS^{-1}V$ is integral) coincides with the space spanned by the theta series $\vartheta_P(Z, Q \,|\, T)$ (Q even, or just integral if q is even, such that qQ^{-1} is even and $T \in T_{n,q}(Q)$).*

We recall moreover that by means of the theta-transformation formulas and from the well known expression of the multiplier systems coming from quadratic forms we also have:

2.5. Proposition *Let $Q = Q^{(r,r)}$ be an even positive definite matrix of level dividing q (i.e., qQ^{-1} is even) and let $T \in T_{n,q}(Q)$. If P is a harmonic form with respect to the polynomial (reduced) representation ρ_0, and $\rho = [\rho_0, r]$, then*

$$\vartheta_P(Z, Q \,|\, T) \in \begin{cases} [\Gamma_n[q], \rho, 1] & \text{if } r \equiv 0 \bmod 2 \\ [\Gamma_n[q], \rho, \chi_{(2)}^{(n)\,r}] & \text{if } r \equiv 1 \bmod 2 \end{cases}$$

(See [**A.Z.**])

2.6. Remark By means of Proposition 2.4 we get that if q is odd the same result as above holds also for theta series $\vartheta_P^{(n)} \left[\begin{array}{c} 0 \\ \frac{V}{q} \end{array} \right] (S/q, Z)$ (with S and q^2S^{-1} even and $V = V^{(r,n)}$ integral such that $S^{-1}[V]$ is even and $qS^{-1}V$ is integral).

The multiplier system $v_{S/q}$ can be expressed as a Gauss sum. In fact if we consider the group $\Gamma^n(S/q)$, which is by definition:

$$\left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R}) \,\Big|\, \begin{pmatrix} A \otimes E & B \otimes S/q \\ C \otimes qS^{-1} & D \otimes E \end{pmatrix} \in \Gamma_{n,\vartheta} \right\},$$

then $\forall M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^n(S/q)$ such that D is invertible it holds

$$v_{S/q}(M) = (\det D)^{-r/2} \sum_{H \in \mathbb{Z}^{(r,n)}/\,{}^tD\mathbb{Z}^{(r,n)}} \exp(\pi i \sigma(BD^{-1}\frac{S}{q}[H]));$$

hence,

$$v_{S/q}(M) = \chi_S^{(n)}(\tilde{M})$$

where $\tilde{M} = \begin{pmatrix} A & B/q \\ qC & D \end{pmatrix} \in \Gamma_{n,0}\left[q^2\right]$, and $\chi_S^{(n)}$ is the multiplier system coming from the Dirichlet character associated with the integral quadratic form S (modulo the level of S). That is:

· in the case of integral weight k we have

$$\chi_S^{(n)}(M) = \chi_S(\det D) := sgn(\det D)^k \left(\frac{discS}{|\det D|}\right);$$

· in the case of half-integral weight $k = \frac{r}{2}$, assuming $\Gamma_{n,0}\left[4\right] \geq \Gamma$,

$$\chi_S^{(n)}(M)\det(CZ+D)^k = \chi_{S\oplus(2)}(M)\chi_{2I_2}\left(M\right)^{(r+1)/2} J_{(2)}^{(n)}(M,Z)^r,$$

with $J_{(2)}^{(n)}(M,Z) = \frac{\vartheta_{(2)}^{(n)}(M\langle Z\rangle)}{\vartheta_{(2)}^{(n)}(Z)}$.

In order to discuss the problem of the extension of the multiplier systems, it is useful, to consider the conductor of the Dirichlet character χ_S. Thus, for the sake of completeness, we recall that:

$$(3) \quad cond\,(\chi_S) = \begin{cases} sfn\,(\det S) & \text{if } (-1)^k sfn\,(\det S) \equiv 1 \bmod 4 \\ 4 \cdot sfn\,(\det S) & \text{otherwise} \end{cases}$$

where by $sfn(m)$ we mean the square free nucleus of the positive integer m (See [**Br.**], [**B.F.S.-P**]).

We end this section recalling the definition of the Siegel's Φ-operator. Let us consider a function $f : \mathbb{H}_n \to \mathcal{Z}$. Then we define

$$\begin{aligned} \Phi(f) &: \quad \mathbb{H}_{n-1} \to \mathcal{Z} \\ \Phi(f)(Z) &:= \quad \lim_{t\to\infty} f\begin{pmatrix} it & 0 \\ 0 & Z \end{pmatrix} \end{aligned}$$

whenever the limit does exist.

For functions f which have a Fourier expansion (convergent on $\mathbb{H}_n$) of the type:

$$\begin{aligned} f(Z) &= \sum_{T\in L^*} a(T)\exp(2\pi i\sigma(TZ)) \\ a(T) \neq 0 &\implies T \geq 0 \end{aligned}$$

the Siegel operator can be applied term by term; thus one obtains

$$\Phi(f)(Z) = \sum_{\begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}\in L^*} a\begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}\exp(2\pi i\sigma(TZ)).$$

In particular if f is a theta series $\vartheta_P^{(n)}\left[\begin{array}{c} U \\ V \end{array}\right](S,Z)$, we have

$$\Phi(\vartheta_P^{(n)}\left[\begin{array}{c} U \\ V \end{array}\right](S)) = \begin{cases} \vartheta_{P_0}^{(n-1)}\left[\begin{array}{c} U_2 \\ V_2 \end{array}\right](S) & \text{if } u_1 = 0; \\ 0 & \text{otherwise,} \end{cases}$$

where $U = (\overbrace{u_1}^{1}, \overbrace{U_2}^{n-1})$, $V = (\overbrace{v_1}^{1}, \overbrace{V_2}^{n-1})$, and $P_0(X) := P(0,X)$.
By applying the Siegel operator on spaces of modular forms $[\Gamma, \rho, v]$ $(\Gamma \leq Sp(n,\mathbb{R}))$
we get maps

$$\Phi : [\Gamma, \rho, v] \to [\Gamma|_\Phi, \rho|_\Phi, v|_\Phi] .$$

We refer to [**Fr.1**] for the general definitions of the congruence subgroup $\Gamma|_\Phi \leq Sp(n-1,\mathbb{R})$, the representation $\rho|_\Phi$, and the multiplier $v|_\Phi$. We just want to add that $\Gamma_n[q]|_\Phi = \Gamma_{n-1}[q]$.

3. The structure of the quotient

We devote this section to the descrption of the quotients we have taken into account, that is $\frac{\Gamma_n[N]}{\Gamma_n[q]}$ $(q = N \cdot p;\ p|N)$ and $\frac{\Gamma_n[N]}{\Gamma_{n+1}[q]}$ for an odd integer $q = N \cdot p$, p prime, $(p, N) = 1$.

3.1. Lemma *$\frac{\Gamma_n[N]}{\Gamma_n[q]} \cong (C_p)^{n(2n+1)}$ $(q = N \cdot p;\ p|N)$.*

Proof. We observe that $\left|\frac{\Gamma_n[N]}{\Gamma_n[q]}\right| = [\Gamma_n : \Gamma_n[q]] \cdot [\Gamma_n : \Gamma_n[N]]^{-1} = p^{n(2n+1)}$. Moreover, two simple and straightforward calculations show that the commutator subgroup $\Gamma_n[N]' \leq \Gamma_n[N^2] \leq \Gamma_n[q]$, and $\forall\ M \in \Gamma_n[N]$ the p-th power $M^p \in \Gamma_n[q]$. These facts imply that our quotient is an elementary abelian p-group. ∎

In order to give a complete set of coset representatives we consider the following matrices:

- $\mathtt{A}_{ij}(\sigma) = \begin{pmatrix} E + N\sigma e_{ij} & 0 \\ 0 & E - N\sigma e_{ji} \end{pmatrix}$ with $i \neq j, 1 \leq i,j \leq n, \sigma \in \mathbb{F}_p$;
- $\mathtt{B}_{ij}(\sigma) = \begin{pmatrix} E & N\sigma e_{ij} + N\sigma e_{ji} \\ 0 & E \end{pmatrix}$ with $1 \leq i,j \leq n, \sigma \in \mathbb{F}_p$;
- $\mathtt{C}_{ij}(\sigma) = {}^t\mathtt{B}_{ij}(\sigma)$ with $1 \leq i,j \leq n, \sigma \in \mathbb{F}_p$;
- $\mathtt{d}_i(\sigma) = \begin{pmatrix} E + N\sigma e_{ii} & -N\sigma e_{ii} \\ N\sigma e_{ii} & E - N\sigma e_{ii} \end{pmatrix}$ with $1 \leq i \leq n, \sigma \in \mathbb{F}_p$;

and we call $\mathtt{A}(p)$, $\mathtt{B}(p)$, $\mathtt{C}(p)$, and $\mathtt{d}(p)$, the subgroups of $\frac{\Gamma_n[N]}{\Gamma_n[q]}$ which are generated by $\mathtt{A}_{ij}(\sigma)$, $\mathtt{B}_{ij}(\sigma)$, $\mathtt{C}_{ij}(\sigma)$ and $\mathtt{d}_i(\sigma)$, respectively (for all possible choices of i, j, and σ); hence, $|\mathtt{A}(p)| = p^{n(n-1)}$, $|\mathtt{B}(p)| = |\mathtt{C}(p)| = p^{n(n+1)/2}$, $|\mathtt{d}(p)| = p^n$ and we easily see that $\frac{\Gamma_n[N]}{\Gamma_n[q]} = \mathtt{d}(p)\times\mathtt{A}(p)\times\mathtt{B}(p)\times\mathtt{C}(p)$. Actually, for our purposes a slightly different description of the representatives will be more useful. In fact, we can multiply every $\mathtt{d}_i(\sigma)$ by suitable elements of $\mathtt{B}(p)$ and $\mathtt{C}(p)$ and obtain a matrix $\delta_i(\sigma) \equiv \begin{pmatrix} E+N\sigma e_{ii} & 0 \\ 0 & E-N\sigma e_{ii}\end{pmatrix} \bmod q$. So, fixing the notation

$$\frac{\Gamma_n\,[N]}{\Gamma_n\,[q]} = \{\delta\cdot\mathtt{a}\cdot\mathtt{b}\cdot\mathtt{c}\}\,, \tag{4}$$

with $\mathtt{a}\in\mathtt{A}(p)$, $\mathtt{b}\in\mathtt{B}(p)$, $\mathtt{c}\in\mathtt{C}(p)$ and δ the product of $\delta_i(\sigma)$'s.

Let us now consider the second quotient group we have to study. Of course if $(p, N) = 1$ the isomorphism $\frac{\Gamma_{n+1}[N]}{\Gamma_{n+1}[q]} \cong Sp(n+1, \mathbb{F}_p)$ holds; but we need a description of this group that can be compatible with our procedure that, as we know, involves the use of the Siegel operator. In order to obtain such a convenient decomposition of $Sp(n+1, \mathbb{F}_p)$, we consider the following two parabolic subgroups:

-) $P_{n+1,0} = P_{n+1} := \left\{\begin{pmatrix} A & B \\ 0 & D\end{pmatrix} \in Sp(n+1, \mathbb{F}_p)\right\}$;

-) $P_{n+1,n} := \left\{\begin{pmatrix} \alpha_1 & \alpha_2 & \beta_1 & \beta_2 \\ 0 & A_4 & \beta_3 & B_4 \\ 0 & 0 & \delta_1 & 0 \\ 0 & C_4 & \delta_3 & D_4\end{pmatrix} \in Sp(n+1, \mathbb{F}_p)\right\}$;

and the relative partition in double cosets

$$Sp(n+1, \mathbb{F}_p) = \bigcup_i P_{n+1} h_i P_{n+1,n}\ .$$

There are actually just two equivalence classes defined by the above partition and we can choose as representatives $\omega_0 := E_{2n}$ and the embedded 1-involution $\omega_1 = \omega_1(p)$. In a standard way we can now decompose each double coset as an union of right cosets. In fact, if we put

$$\begin{aligned} \bar{P}^0 &= P_{n+1}/P_{n+1}\cap P_{n+1,n}\ ; \\ \bar{P}^1 &= P_{n+1}/P_{n+1}\cap \omega_1 P_{n+1,n}\omega_1^{-1}, \end{aligned}$$

then

$$\begin{aligned} P_{n+1}P_{n+1,n} &= \bar{P}^0 P_{n+1,n}\ ; \\ P_{n+1}\omega_1 P_{n+1,n} &= \bar{P}^1\omega_1 P_{n+1,n}\ . \end{aligned} \tag{5}$$

Since $P_{n+1} \cap P_{n+1,n} = \{M \in P_{n+1,n} \mid C_4 = 0\}$, we see that a possible explicit description of the $(p^{n+1}-1)/(p-1)$ elements of $\bar{P}^0$ can be given considering the matrices:

-) $m^0(x_1) = \begin{pmatrix} 1 & 0 \\ x_1 & E_{(n)} \end{pmatrix}$, $x_1 \in \mathbb{F}_p^n$;

$\vdots$

-) $m^0(x_h) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & E_{(h-2)} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ x_h & 0 & 0 & E_{(n+1-h)} \end{pmatrix}$, $x_h \in \mathbb{F}_p^{n+1-h}$;

$\vdots$

-) $m^0(1) = m^0(x_{n+1}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & E_{(n-1)} & 0 \\ 1 & 0 & 0 \end{pmatrix}$;

and completing each of them to a symplectic matrix

$$p^0(x_h) = \begin{pmatrix} m^0(x_h) & 0 \\ 0 & {}^t m^0(x_h)^{-1} \end{pmatrix}.$$

On the other hand, it turns out that

$$P_{n+1} \cap \omega_1 P_{n+1,n} \omega_1^{-1} = \left\{ \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ \alpha_3 & A_4 & 0 & B_4 \\ 0 & 0 & \delta_1 & \delta_3 \\ 0 & 0 & 0 & D_4 \end{pmatrix} \in Sp(n+1, \mathbb{F}_p) \right\},$$

and so

$$\bar{P}^1 = \{p^1(x_h),\ 1 \le h \le n+1\} \times \left\{ \begin{pmatrix} 1 & 0 & \beta_1 & \beta_2 \\ 0 & E_{(n)} & {}^t\beta_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & E_{(n)} \end{pmatrix} \in Sp(n+1, \mathbb{F}_p) \right\},$$

where, with notation similar to those we have just used for $\bar{P}^0$, we define

$$m^1(x_h) := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & E_{(h-2)} & 0 & 0 \\ 1 & 0 & 0 & x_h \\ 0 & 0 & 0 & E_{(n+1-h)} \end{pmatrix},\ x_h \in \mathbb{F}_p^{n+1-h},$$

and then

$$p^1(x_h) := \begin{pmatrix} m^1(x_h) & 0 \\ 0 & {}^t m^1(x_h)^{-1} \end{pmatrix}.$$

4. The action of the trace operator on theta series and the commutation formula

We come now to the main part of our work. It turns out that there are two different basic cases that we shall treat separately.

A) From $\mathbf{\Gamma_n\,[N\cdot p]}$ to $\mathbf{\Gamma_n\,[N]}$ with $\mathbf{p|N}$

We take into account the vector space spanned by the theta series $\vartheta_P^{(n)}\left[\begin{array}{c}0\\ \frac{V}{q}\end{array}\right]\left(\frac{S}{q},Z\right)$ in $\left[\Gamma_n\,[q]\,,\rho,v_{\frac{S}{q}}\right]$. We know that we can restrict ourself to consider even quadratic forms S of level q, if q is odd, or such that qS^{-1} is integral, if q is even (see Remark 2.3). From now on we shall assume these restrictions on the level of S. Moreover, we treat first the case of integral weight.

We consider a Dirichlet character χ_N on $\Gamma_n\,[N]$, acting in the usual manner, that extends $v_{\frac{S}{q}}$, i.e.:

$$\chi_N|_{\Gamma_n[q]} = v_{\frac{S}{q}}|_{\Gamma_n[q]}. \tag{6}$$

Since we are going to need the results on the theory of singular modular forms on $\Gamma_n\,[N]$ we have to assume that the extended character χ_N satisfies the Freitag's condition (see [**Fr.1**]).

So, referring for the cosets representatives to the list we have produced after Lemma 3.1, we introduce the trace operator:

$$Tr^{\Gamma_n[q],v_{\frac{S}{q}}}_{\Gamma_n[N],\chi_N}\vartheta_P^{(n)}\left[\begin{array}{c}0\\ \frac{V}{q}\end{array}\right]\left(\frac{S}{q},Z\right) = \\ = \sum_{\delta\cdot\mathsf{a}\cdot\mathsf{b}\cdot\mathsf{c}\in\frac{\Gamma_{n,0}[q]}{\Gamma_n[q]}} \chi_N^{-1}(\delta)\,\vartheta_P^{(n)}\left[\begin{array}{c}0\\ \frac{V}{q}\end{array}\right]\left(\frac{S}{q},Z\right)\,|_{k,\rho}\,\delta\cdot\mathsf{a}\cdot\mathsf{b}\cdot\mathsf{c}\,.$$

For the rest of this subsection we shall shorten the previous expression as: $Tr^{(n)}\vartheta_{P,[V]}\,(S,Z)$.

Let $\vartheta_{P_1}^{(n+1)}\left[\begin{array}{cc}0&0\\0&\frac{V}{q}\end{array}\right]\left(\frac{S}{q},Z\right)\in\left[\Gamma_{n+1}\,[q]\,,\rho_1,v_{\frac{S}{q}}\right]$ be the lifted series for which

$$\Phi(\vartheta_{P_1}^{(n+1)}\left[\begin{array}{cc}0&0\\0&\frac{V}{q}\end{array}\right]\left(\frac{S}{q},Z\right)) = \vartheta_P^{(n)}\left[\begin{array}{c}0\\ \frac{V}{q}\end{array}\right]\left(\frac{S}{q},Z\right)$$

holds.

In what follows we shall sometimes make use of following two "rules", (See [**B.F.S.-P.**]):

$$\mathbf{r_1})\ \Phi(f|_{\rho_1}\begin{pmatrix} 1 & a_2 & b_1 & b_2 \\ 0 & E_n & {}^t b_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -{}^t a_2 & E_n \end{pmatrix}) = \Phi(f)\ ;$$

$$\mathbf{r_2})\ \Phi(f|_{\rho_1}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & 1 & 0 \\ 0 & C & 0 & D \end{pmatrix}) = \Phi(f)\,|_{\rho}\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

.

(In these "rules" the matrices are in $Sp(n,\mathbb{R})$ and f is a Fourier series converging on $\mathbb{H}_{n+1}$, whose coefficients depend on some lattice of positive semi-definite symmetric matrices).

4.1. Proposition $\Phi(Tr^{(n+1)}\vartheta_{P_1,[0,V]}(S,Z)) = K\cdot Tr^{(n)}\vartheta_{P,[V]}(S,Z)\,)$ *($K = K(p,n,S,V)$, positive constant).*

Proof. From rules $\mathbf{r_1})$, $\mathbf{r_2})$, we get

$$\Phi(Tr^{(n+1)}\vartheta_{P_1,[0,V]}(S,Z))$$
$$= p^{2n+1}Tr_\epsilon^{(n)}\Phi(\sum_{\delta,\mathtt{a}_3,\mathtt{c}_1,\mathtt{c}_2}\chi_N^{-1}(\delta)\,\vartheta_{P_1,[0,V]}(S,Z)\ |_{\rho_1}\delta\cdot\mathtt{m}(\mathtt{a}_3)\cdot\mathtt{m}(\mathtt{c}_1,\mathtt{c}_2)\,)$$

where $Tr_\epsilon^{(n)}$ is the trace restricted to the $\mathtt{a}\cdot\mathtt{b}\cdot\mathtt{c}$ products and

$$\mathtt{m}(\mathtt{a}_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \mathtt{a}_3 & E & 0 & 0 \\ 0 & 0 & 1 & -{}^t\mathtt{a}_3 \\ 0 & 0 & 0 & E \end{pmatrix},\ \mathtt{m}(\mathtt{c}_1,\mathtt{c}_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & E_n & 0 & 0 \\ \mathtt{c}_1 & \mathtt{c}_2 & 1 & 0 \\ {}^t\mathtt{c}_2 & 0 & 0 & E_n \end{pmatrix}$$

($\mathtt{m}(\mathtt{a}_3) \in \mathtt{A}(p)$, $\mathtt{m}(\mathtt{c}_1,\mathtt{c}_2) \in \mathtt{C}(p)$).

By definition, δ, $\mathtt{m}(\mathtt{a}_3)$ and $\mathtt{m}(\mathtt{c}_1,\mathtt{c}_2)$ belong to the group $\Gamma\left(\frac{S}{q}\right)$ (here, for $\mathtt{m}(\mathtt{c}_1,\mathtt{c}_2)$, is important the restriction on the level of S). Hence, decomposing each δ as $\delta \equiv \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_4 & 0 & 0 \\ 0 & 0 & \delta_1 & 0 \\ 0 & 0 & 0 & \delta_4 \end{pmatrix} \bmod q$, and using theta-transformation formulas (see [**Fr.1**]), yields

$$\sum_{\delta,\mathtt{a}_3,\mathtt{c}_1,\mathtt{c}_2,}\chi_N^{-1}(\delta)\,\vartheta_{P_1,[0,V]}(S,Z)\ |_{\rho_1}\delta\cdot\mathtt{m}(\mathtt{a}_3)\cdot\mathtt{m}(\mathtt{c}_1,\mathtt{c}_2)\ =$$
$$= p\sum_{\delta_4,\mathtt{a}_3,\mathtt{c}_1,\mathtt{c}_2}\vartheta_{P_1,[0,V\delta_4]}(S,Z)\ |_{\rho_1}\mathtt{m}(\mathtt{a}_3)\cdot\mathtt{m}(\mathtt{c}_1,\mathtt{c}_2) =$$

Here, we have assumed that $\chi_N(\delta) = v_{\frac{S}{q}}(\delta)$ to avoid the vanishing of the previous expression. We point out that, in view of (3) and (6), this is always true unless we are dealing with the prime $p = 2$. Carrying on our computation we get:

$$\cdots = p^{n+1} \sum_{\delta_4,\mathtt{c}_1,\mathtt{c}_2} \vartheta_{P_1,[0,V\delta_4]}(S,Z) \,|_{\rho_1}\mathtt{m}(\mathtt{c}_1,\mathtt{c}_2) =$$

$$= p^{n+2} \sum_{\delta_4,\mathtt{c}_2} \vartheta_{P_1}^{(n+1)} \begin{bmatrix} S^{-1}V\delta_4{}^t\mathtt{c}_2 & 0 \\ 0 & \frac{V\delta_4}{q} \end{bmatrix} (S,Z)\,.$$

In conclusion, applying Siegel's operator, and using that $qS^{-1}V$ is integral gives

$$\Phi(Tr^{(n+1)}\vartheta_{P_1,[0,V]}(S,Z)) = k(p)Tr_\epsilon^{(n)} \sum_{\delta_4} \sum_{S^{-1}V\mathtt{c}_2 \text{ integral}} \vartheta_{P,[V\delta_4]}(S,Z) =$$

$$= K(p,n,S,V)Tr^{(n)}\vartheta_{P,[V]}(S,Z)\,. \quad \blacksquare$$

From the commutation formula the theorem follows on the representability of the image of the trace operator as combination of appropriate theta series, since we can apply again the same procedure h times, with h such that $n+h$ is large enough to ensure that every theta series of that degree is singular, and conclude that:

$$\Phi^{(h)}(Tr^{(n+h)}\vartheta_{P_h,[0,\cdots,0,V]}(S,Z)) = cTr^{(n)}\vartheta_{P,[V]}(S,Z)\,, \tag{7}$$

where c is some positive constant. (From this relation the thesis follows once we have acted on the left hand side with Siegel's operator)

4.2. Theorem *Let $r = 2k$ be a positive even integer, and ρ_0 a rational reduced representation of $GL(n,\mathbb{C})$, $\rho = [\rho_0, r]$. Fix the integer $q = N \cdot p$, with p prime such that $p|N$ and call $\Theta\left[\Gamma_n[N]\right]_{\chi_N,\rho} \subset [\Gamma_n[N],\rho,\chi_N]$ the vector space spanned by the theta series of weight k of type $\vartheta_P^{(n)}\begin{bmatrix} 0 \\ \frac{V}{N}\end{bmatrix}\left(\frac{S}{N},Z\right)$. Let $\left[\Gamma_n[q],\rho,v_{\frac{S}{q}}\right] \supset [\Gamma_n[N],\rho,\chi_N]$ and consider $\Theta\left[\Gamma_n[q]\right]_{\chi_N,\rho} \subset \left[\Gamma_n[q],\rho,v_{\frac{S}{q}}\right]$, the vector space spanned by the theta series of weight k of type $\vartheta_P^{(n)}\begin{bmatrix} 0 \\ \frac{V}{q}\end{bmatrix}\left(\frac{S}{q},Z\right)$, such that $\chi_N|_{\Gamma_0^0[q]\cap\Gamma[N]} = v_{\frac{S}{q}}|_{\Gamma_0^0[q]\cap\Gamma[N]}$ (this is always the case for $p \neq 2$); assume finally that χ_N satisfies Freitag's conditions (see* [**Fr.1**]*), then:*

$$Tr_{\Gamma_n[N],\chi_N}^{\Gamma_n[q],v_{\frac{S}{q}}} \vartheta_P^{(n)}\begin{bmatrix} 0 \\ \frac{V}{q}\end{bmatrix}\left(\frac{S}{q},Z\right) \in \Theta\left[\Gamma_n[N]\right]_{\rho,\chi_N}\,.$$

Therefore, we also have:

$$\Theta\left[\Gamma_n[q]\right]_{\rho,\chi_N} \cap [\Gamma_n[N],\rho,\chi_N] = \Theta\left[\Gamma_n[N]\right]_{\rho,\chi_N}\,.$$

4.3. Remark Up to now we have considered only integral weights. However we notice that if S is an integral quadratic form of level (dividing) $2q$, in an odd number of variables (hence $2q \equiv 0 \bmod 4$), we can consider the theta series

$$\vartheta_{P,[V]}\left(2+S,Z\right) := \vartheta^{(n)}\left(2,Z\right)\cdot\vartheta_P^{(n)}\left[\begin{array}{c}0\\ \frac{V}{q}\end{array}\right]\left(\frac{S}{q},Z\right).$$

Since one checks that

$$Tr^{(n)}\vartheta_{P,[V]}\left(2+S,Z\right) = \vartheta^{(n)}\left(2,Z\right)Tr^{(n)}\vartheta_{P,[V]}\left(S,Z\right)$$

the case of half-integral weights is obtained as a consequence of the above discussion.

We would like moreover to notice that with exactly the same technique, it is possible to prove an analogous result for the Igusa subgroups $\Gamma_n\left[q,2q\right]$ and $\Gamma_n\left[N,2N\right]$ (with $p|N$).

B) From $\mathbf{\Gamma_n[q]}$ to $\mathbf{\Gamma_n[N]}$ with $\mathbf{q = pN}$ and $\mathbf{(p,N) = 1}$

In this final subsection we are going to present our results regarding the trace operator on theta series for principal congruence subgroups in the case of the change of level from $\Gamma_n[q]$ to $\Gamma_n\left[N\right]$, for $q = N\cdot p$ with p prime and $(N,p) = 1$. For the sake of simplicity we will not discuss half integral weights; these can be however handled by means of reasonings such as in Remark 4.3.

We want to consider first the simplest case i.e. $q = p$ an odd prime. Since we deal with an odd prime, we cannot allow the weight to be half-integral. So, for even positive definite quadratic forms $S = S^{(r,r)}$ (r even) of level p, we consider the theta series $\vartheta_P^{(n)}\left(Z,S\,|\,T\right)\in\left[\Gamma_n\left[p\right],\rho,1\right]$. $S[T]\equiv 0 \bmod p^2$ (this is not a restriction, since otherwise the trace would Then, we are to deal with

$$Tr_{\Gamma_{n},1}^{\Gamma_n[p],1}\vartheta_P^{(n)}\left(Z,S\,|\,T\right) = \sum_{\gamma\in\frac{\Gamma_n}{\Gamma_n[p]}}\vartheta_P^{(n)}\left(Z,S\,|\,T\right)\,|_{k,\rho}\,\gamma$$

or rather, we want to evaluate

$$\Phi(Tr_{\Gamma_{n+1},1}^{\Gamma_{n+1}[p],1}\vartheta_{P_1}^{(n+1)}(Z,S\,|\,[0,T])) \tag{8}$$

and try to establish a commutation formula.

Before beginning to evaluate (8) we remark that from now on the characteristic T will be assumed to be in canonical form, that is

$$T = [t_1,...,t_\iota,0,...,0],\ rank(T)_{\mathbb{F}_p} = \iota\ . \tag{9}$$

We fix for the moment $\iota > 0$. Moreover, we assume that $S[T]\equiv 0 \bmod p^2$ (this is not a restriction, since otherwise the trace would vanish, as follows from the computation of the trace operator restricted to the subgroup of the translations $T(B)$).

Using the decomposition (5) and then applying rule $\mathbf{r}_1$, and denoting $m(\alpha_1)$ a representative in Γ_{n+1} of the matrix $\begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & \alpha_1^{-1} & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \in Sp(n+1, \mathbb{F}_p)$, we immediately get the following identity

(10)
$$\Phi(Tr^{\Gamma_{n+1}[p],1}_{\Gamma_{n+1},1} \vartheta^{(n+1)}_{P_1}(Z, S \mid [0,T])) =$$
$$= p^{2n+1} Tr^{\Gamma_n[p],1}_{\Gamma_n,1} \Phi(\sum_{\alpha_1 \in \mathbb{F}_p^*} \sum_{\gamma \in \bar{P}^0 \cup \bar{P}^1 \omega_1} \vartheta^{(n+1)}_{P_1}(Z, S \mid [0,T]) \mid_{k,\rho_1} \gamma m(\alpha_1)).$$

In order to get some commutation formula we have at least to avoid the vanishing of the l.h.s. of (10), hence, taking into account the action of the subgroup generated by the $m(\alpha_1)$'s, we are led to impose the further condition $v_S|_{\Gamma_0^0(p)} = 1$. In particular this implies that $r \equiv 0 \bmod 4$ and $\det S = p^{2l}$ (l a non-negative integer).
We extract from (10) the two contributions:

$$\sum_{\alpha_1 \in \mathbb{F}_p^*} \sum_{\gamma \in \bar{P}^l \omega_l} \Phi(\vartheta^{(n+1)}_{P_1}(Z, S \mid [0,T]) \mid_{k,\rho_1} \gamma m(\alpha_1)) \ (l = 0, 1).$$

We expand the first one ($l = 0$) using the above description of $\bar{P}^0$

$$\sum_{\alpha_1 \in \mathbb{F}_p^*} \sum_{\gamma \in \bar{P}^0} \Phi(\vartheta^{(n+1)}_{P_1}(Z, S \mid [0,T]) \mid_{k,\rho_1} \gamma m(\alpha_1)) =$$
$$= \sum_{\alpha_1 \in \mathbb{F}_p^*} \sum_{1 \le h \le n+1} \sum_{x_h \in \mathbb{F}_p^{n+1-h}} \Phi(\vartheta^{(n+1)}_{P_1}(Z, S \mid [0,T]) \mid_{k,\rho_1} p^0\,(x_h)\, m(\alpha_1)) =$$
$$= \sum_{\alpha_1 \in \mathbb{F}_p^*} \sum_{x_1 \in \mathbb{F}_p^n} \Phi(\vartheta^{(n+1)}_{P_1}(Z, S \mid [Tx_1, T\,]))+$$
$$+ \sum_{x_h \in \mathbb{F}_p^{n-1}} \Phi(\vartheta^{(n+1)}_{P_1}(Z, S \mid [\alpha_1 t_1 + T_2 x_2, 0, T_3]))+$$
$$+ \cdots + \sum_{x_{\iota+1} \in \mathbb{F}_p^{n-\iota}} \Phi(\vartheta^{(n+1)}_{P_1}(Z, S \mid [\alpha_1 t_\iota, t_1, \ldots, t_{\iota-1}, 0, \ldots, 0]))+$$
$$+ \sum_{\iota+2 \le h \le n+1} \sum_{x_h \in \mathbb{F}_p^{n+1-h}} \Phi(\vartheta^{(n+1)}_{P_1}(Z, S \mid [0,T]))$$

(with the notation $T = [t_1, \cdots, t_k, T_{k+1}]$).
The above sum, applying Siegel's operator, becomes:

$$= (p-1)(p^{n-\iota} \vartheta^{(n)}_P\,(Z, S \mid T) + 0 + \cdots + 0+$$
$$+ (p^{n-\iota-1} + \cdots + p + 1) \vartheta^{(n)}_P\,(Z, S \mid T))$$

Hence, for the contribution relative to $\bar{P}^0$ we have:

$$\sum_{\gamma\in\bar{P}^0}\Phi(\vartheta_{P_1}^{(n+1)}(Z,S\,|\,0,T)\ |_{k,\rho_1}\gamma)=(p^{n-\iota+1}-1)\vartheta_P^{(n)}(Z,S\,|\,T)\,. \tag{11}$$

Now let us deal with the $\bar{P}^1\omega_1$ sum.

$$\sum_{\alpha_1\in\mathbb{F}_p^*}\sum_{\gamma\in\bar{P}^1\omega_1}\Phi(\vartheta_{P_1}^{(n+1)}(Z,S\,|\,[0,T])\,|_{k,\rho_1}\gamma m(\alpha_1))=$$

$$=\sum_{\alpha_1\in\mathbb{F}_p^*}\sum_{(\beta_1,\beta_2)\in\mathbb{F}_p^{n+1}}\sum_{1\le h\le n+1}\sum_{x_h\in\mathbb{F}_p^{n+1-h}}\Phi(\vartheta_{P_1}^{(n+1)}(Z,S\,|\,[0,T])\,|_{k,\rho_1}p^1\,(x_h)\cdot$$

$$\cdot T(\begin{pmatrix}\beta_1 & \beta_2\\ {}^t\beta_2 & 0\end{pmatrix})m(\alpha_1)\omega_1)=$$

$$=\Phi(\sum_{\alpha_1\in\mathbb{F}_p^*}\sum_{(\beta_1,\beta_2)\in\mathbb{F}_p^{n+1}}(\sum_{x_1\in\mathbb{F}_p^n}\vartheta_{P_1}^{(n+1)}(Z,S\,|\,[0,T])+$$

$$+\sum_{x_2\in\mathbb{F}_p^{n-1}}\vartheta_{P_1}^{(n+1)}(Z,S\,|\,[t_1,0,t_1x_2+T_2])+\cdots+$$

$$+\sum_{x_{k+1}\in\mathbb{F}_p^{n-k}}\vartheta_{P_1}^{(n+1)}(Z,S\,|\,[t_k,t_1,\cdots,t_{k-1},0,t_kx_{k+1}+T_{k+1}])+\cdots+$$

$$+\sum_{x_{\iota+1}\in\mathbb{F}_p^{n-\iota}}\vartheta_{P_1}^{(n+1)}(Z,S\,|\,[t_\iota,t_1,\cdots,t_{\iota-1},0,t_\iota x_{\iota+1}])+$$

$$+\sum_{\iota+2\le h\le n+1}\sum_{x_h\in\mathbb{F}_p^{n+1-h}}\vartheta_{P_1}^{(n+1)}(Z,S\,|\,[0,T]))\,|_{k,\rho_1}T(\begin{pmatrix}\beta_1 & \beta_2\\ {}^t\beta_2 & 0\end{pmatrix})m(\alpha_1)\omega_1).$$

By using transformation formulas for $T(\begin{pmatrix}\beta_1 & \beta_2\\ {}^t\beta_2 & 0\end{pmatrix})$ we have:

$$\cdots=\Phi(\sum_{\alpha_1\in\mathbb{F}_p^*}\sum_{(\beta_1,\beta_2)\in\mathbb{F}_p^{n+1}}p^n\exp(\tfrac{\pi i}{p^2}\sigma(S[[0,T]]\begin{pmatrix}\beta_1 & \beta_2\\ {}^t\beta_2 & 0\end{pmatrix}))\vartheta_{P_1}^{(n+1)}(Z,S\,|\,[0,T])+$$

$$+\sum_{2\le k\le\iota+1}\sum_{x_k}\exp(\tfrac{\pi i}{p^2}\sigma(S[[t_{k-1},t_1,\cdots,t_{k-2},0,t_{k-1}x_k+T_k]]\begin{pmatrix}\beta_1 & \beta_2\\ {}^t\beta_2 & 0\end{pmatrix}))\cdot$$

$$\cdot\vartheta_{P_1}^{(n+1)}(Z,S\,|\,[0,T]m^1(x_k))+\tfrac{p^{n-\iota}-1}{p-1}\exp(\tfrac{\pi i}{p^2}\sigma(S[[0,T]]\begin{pmatrix}\beta_1 & \beta_2\\ {}^t\beta_2 & 0\end{pmatrix}))\cdot$$

$$\cdot\vartheta_{P_1}^{(n+1)}(Z,S\,|\,[0,T])\,|_{k,\rho_1}m(\alpha_1)\omega_1)=$$

$$=\Phi(p^{2n+1}(p-1)\vartheta_{P_1}^{(n+1)}(Z,S\,|\,[0,T])\,|_{k,\rho_1}\omega_1+$$

$$+(p^{n+1}\sum_{\alpha_1\in\mathbb{F}_p^*}\sum_{2\le k\le\iota+1}\sum_{x_k}\vartheta_{P_1}^{(n+1)}(Z,S\,|\,[0,T]m^1(x_k))|_{k,\rho_1}m(\alpha_1)+$$

$$+p^{n+1}(p^{n-\iota}-1)\vartheta_{P_1}^{(n+1)}(Z,S\,|\,[0,T]))\,|_{k,\rho_1}\omega_1)$$

(we recall that we have assumed $S[T]\equiv 0\bmod p^2$).

We can now use the formula for the standard symplectic (1-)involution and then apply the Siegel operator so that the previous expression becomes

$$\begin{aligned} &p^{n+1}(\det S)^{-\frac{1}{2}}(i)^{-\frac{r}{2}}((p^n(p-1)+(p^{n-\iota}-1))\vartheta_P^{(n)}(Z,S\,|\,T)+\\ &+\sum_{2\le k\le \iota+1}\sum_{x_k\in\mathbb{F}_p^{n+1-k}}\vartheta_P^{(n)}(Z,S\,|\,[t_1,\cdots,t_{k-2},0,t_{k-1}x_k+T_k]))\,. \end{aligned} \tag{12}$$

Concerning the sums in the second row of (12) we notice that each of them can be decomposed as follows:

$$\sum_{x_k\in\mathbb{F}_p^{n-k+1}}\vartheta_P^{(n)}(Z,S\,|\,[t_1,\cdots,t_{k-2},0,t_{k-1}x_k+T_k]) = \sum_{j=1}^{p^{\iota-k+1}}\vartheta_P^{(n)}(Z,S\,|\,\mathcal{T}_j)+$$
$$+\sum_{j=1}^{p^{\iota-k+1}(p^{n-\iota}-1)}\vartheta_P^{(n)}(Z,S\,|\,\tilde{T}_j),$$

where, for each j, $rank(\mathcal{T}_j)_{\mathbb{F}_p}<\iota$, $(\iota>0)$, whereas for $\tilde{T}_j$ holds:
$Tr_{\Gamma_n,1}^{\Gamma_n[p],1}\vartheta_P^{(n)}(Z,S\,|\,T)=Tr_{\Gamma_n,1}^{\Gamma_n[p],1}\vartheta_P^{(n)}(Z,S\,|\,\tilde{T}_j)$.

Summing up, for the $\bar{P}^1\omega_1$ contribution we have obtained:

(13)

$$\begin{aligned} &p^{2n+1}Tr_{\Gamma_n,1}^{\Gamma_n[p],1}(\Phi(\sum_{\gamma\in\bar{P}^1\omega_1}\sum_{\alpha_1\in\mathbb{F}_p^*}\vartheta_{P_1}^{(n+1)}(Z,S\,|\,[0,T])\,|_{k,\rho_1}\,\gamma m(\alpha_1)))=\\ &=p^{3n+2}(\det S)^{-\frac{1}{2}}(i)^{-\frac{r}{2}}(Tr_{\Gamma_n,1}^{\Gamma_n[p],1}(p^{n+1}-p^\iota)\vartheta_P^{(n)}(Z,S\,|\,T)+\\ &+\sum_{2\le k\le\iota+1}\sum_{j_k=1}^{p^{\iota-k+1}}(p-1)Tr_{\Gamma_n,1}^{\Gamma_n[p],1}(\vartheta_P^{(n)}(Z,S\,|\,\mathcal{T}_{j_k})))\,, \end{aligned}$$

By putting together (11) and (13) we finally achieve the commutation formula:

(14)

$$\begin{aligned} &\Phi(Tr_{\Gamma_{n+1},1}^{\Gamma_{n+1}[p],1}\vartheta_{P_1}^{(n+1)}(Z,S\,|\,[0,T]))=p^{2n+1}Tr_{\Gamma_n,1}^{\Gamma_n[p],1}(((p^{n-\iota+1}-1)+\\ &\qquad+(\det S)^{-\frac{1}{2}}(i)^{-\frac{r}{2}}p^{n+1}(p^{n+1}-p^\iota))\vartheta_P^{(n)}(Z,S\,|\,T))+\\ &+\sum_{2\le k\le\iota+1}\sum_{j_k=1}^{p^{\iota-k+1}}p^{3n+2}(p-1)(\det S)^{-\frac{1}{2}}(i)^{-\frac{r}{2}}Tr_{\Gamma_n,1}^{\Gamma_n[p],1}(\vartheta_P^{(n)}(Z,S\,|\,\mathcal{T}_{j_k}))\,. \end{aligned}$$

On the other hand, in the case $\iota=0$, it is easy to compute the following commutation formula:

$$\begin{aligned} &\Phi(Tr_{\Gamma_{n+1},1}^{\Gamma_{n+1}[p],1}((\vartheta_{P_1}^{(n+1)}(Z,S)))=\\ &=p^{2n+1}(p^{n+1}-1)Tr_{\Gamma_n,1}^{\Gamma_n[p],1}((1+(\det S)^{-\frac{1}{2}}(i)^{-\frac{r}{2}}p^{n+1})\vartheta_P^{(n)}(Z,S)\,, \end{aligned}$$

so that we recover the same result of **[B.F.S.-P.]** (in the case $q=p$). As there, from the last formula we see that (in the case of homogeneous theta series) the only obstruction to the usual procedure is given by the vanishing of the coefficient $1+(\det S)^{-\frac{1}{2}}(i)^{-\frac{r}{2}}p^{\mu+1}$ (for some $n \le \mu < r$).

In the case $\iota > 0$, we have to look to (14). The coefficient of $\vartheta_P^{(n)}(Z, S \mid T))$ vanishes if and only if

$$1 + (\det S)^{-\frac{1}{2}}(i)^{-\frac{r}{2}}p^{n+1+\iota} = 0.$$

Of course this gives an obstruction to the usual recursive procedure. Other obstructions of the same type will be given by the residual terms $\vartheta_P^{(n)}(Z, S \mid \mathcal{T}_j)$, which by construction will satisfy commutation formulas as (14). By going on in the same way till the rank of the residual term is 0, we get the following system of conditions:

$$(15) \qquad 1 + (\det S)^{-\frac{1}{2}}(i)^{-\frac{r}{2}}p^{\mu+1+\kappa} \ne 0 \ \forall n \le \mu < r, \ \forall 0 \le \kappa \le \iota.$$

Let us summarize the previous discussion in the following statement.

4.4. Theorem *Let $r = 2k$, k a even positive integer, and let ρ_0 be a rational reduced representation of $GL(n, \mathbb{C})$, $\rho = [\rho_0, r]$. Call $\Theta[\Gamma_n]_\rho \subset [\Gamma_n, \rho, 1]$ the vector space spanned by the theta series of the type $\vartheta_P^{(n)}(Q, Z)$ and consider $\vartheta_P^{(n)}(Z, S \mid T) \in [\Gamma_n[p], \rho, 1]$ such that* $\det S = p^{2l}$. *If condition (15) is satisfied then*

$$Tr_{\Gamma_n,1}^{\Gamma_n[p],1}\vartheta_P^{(n)}(Z, S \mid T) \in \Theta[\Gamma_n]_\rho .$$

4.5. Remark If $r \equiv 0 \bmod 8$ (that is the only case for which $\Theta[\Gamma_n]_\rho \ne 0$) condition (15) is automatically satisfied, hence, if we let the vector space spanned by the above theta series $\vartheta_P^{(n)}(Z, S \mid T)$ be denoted by $\Theta[\Gamma_n[p]]_{\rho,1}$,

$$\Theta[\Gamma_n[p]]_\rho \cap [\Gamma_n, \rho, 1] = \Theta[\Gamma_n]_\rho .$$

The condition is therefore significant just in the case $r \equiv 4 \bmod 8$, but in fact its fulfillment implies the vanishing of the trace.
It is also maybe worthwhile repeating that, even if $r \equiv 2 \bmod 4$, the trace vanishes if $S[T]$ is not $0 \bmod p^2$.

We want now to analize the more general situation $q = N \cdot p$ with $(N, p) = 1$ for an odd q. So we consider theta series $\vartheta_P^{(n)}(Z, S \mid T) \in [\Gamma_n[q], \rho, 1]$ with $S = S^{(r,r)}$ $(r = 2k)$, of level q, such that v_S has conductor dividing N (in particular the p-part of $\det S$: $\det_p(S)$, is a square). Moreover we assume that $S[T] \equiv 0 \bmod qp$ (otherwise the trace would vanish).

We fix two integers a and b such that $ap+bN = 1$ and we consider the integer γ defined by means of the identity $(ap+bN)^N = (ap)^N + bN\gamma =$

1. Hence γ satisfies: $\gamma \equiv 0 \bmod N$ and $\gamma \equiv 1 \bmod p$. Then we can decompose as follows the matrix $\breve{\omega}_1$:

(16)

$$\breve{\omega}_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & bN \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \gamma-(ap)^{N-1} \\ 0 & 1 \end{pmatrix}.$$

In fact it is straightforward to verify that the product in the right hand side of the previous expression is congruent to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N$ and to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bmod p$. Hence we get the following expression for the embedded symplectic 1-involution:

$$\omega_1 = \breve{\omega}_1^{\uparrow} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{\uparrow} := \begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & E_n & 0 & 0 \\ \gamma & 0 & \delta & 0 \\ 0 & 0 & 0 & E_n \end{pmatrix} \in \Gamma_{n+1}[N]$$

which allows us to write the identity:

$$\Phi(\vartheta^{(n+1)}_{P_1}(Z,S\,|\,t_0,T)\ |_{k,\rho_1}\,\omega_1) = \Phi(\vartheta^{(1)}(Z,S\,|\,t_0)\ |_k\,\breve{\omega}_1)\vartheta^{(n)}_P(Z,S\,|\,T).$$

4.6. Lemma *Let $s_p(\mathbb{Q}^r)$ denote the Witt invariant of the completion $\mathbb{Q}^r{}_p$ (normalized as in* [**Sch.**]*), and let $\det_p(S)$ be the p-part of $\det S$ (hence $\det_p(S) = p^{2l}$), then for $t_0 \in T_{1,q}(S)$:*

$$\Phi(\vartheta^{(1)}(Z,S\,|\,t_0)\ |_k\,\breve{\omega}_1) = \begin{cases} s_p(\mathbb{Q}^r)\det_p(S)^{-\frac{1}{2}} & \textit{if } t_0 \equiv 0 \bmod N; \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We want to calculate the value of $\Phi(\vartheta^{(1)}(Z,S\,|\,t_0)\ |_k\,\breve{\omega}_1)$, by using the above decomposition of $\breve{\omega}_1$. We shall refer to the five factors in (16) as $\breve{\omega}_1^{(h)}$, $h = 1,..,5$. By means of the standard theta transformation formulas (see [**A.Z.**]), we get the following identities:

$$\Phi(\vartheta^{(1)}(Z,S\,|\,t_0)\ |_k\,\breve{\omega}_1) =$$

$$= \exp(\pi i q^{-2}S[t_0])\Phi(\vartheta^{(1)}(Z,S\,|\,t_0)\ |_k \textstyle\prod_{h\geq 2}\breve{\omega}_1^{(h)}) =$$

$$= \exp(\pi i q^{-2}S[t_0])(\det S)^{-\frac{1}{2}} i^{-\frac{r}{2}}\Phi(\sum_{t\in T_{1,q}(S)}\exp(2\pi i q^{-2}\,{}^t t_0 S t)\cdot$$

$$\cdot\vartheta^{(1)}(Z,S\,|\,t)\ |_k \textstyle\prod_{h\geq 3}\breve{\omega}_1^{(h)}) =$$

$$= \exp(\pi i q^{-2}S[t_0])(\det S)^{-\frac{1}{2}} i^{-\frac{r}{2}}\Phi(\sum_{t\in T_{1,q}(S)}\exp(\pi i q^{-2}(S[t]bN+$$

$$+2\,{}^t t_0 S t))\cdot\vartheta^{(1)}(Z,S\,|\,t)\ |_k \textstyle\prod_{h\geq 4}\breve{\omega}_1^{(h)}) =$$

$$= \exp(\pi i q^{-2}S[t_0])(\det S)^{-1} i^{-r}\Phi(\sum_{t,t'\in T_{1,q}(S)}\exp(\pi i q^{-2}(S[t]bN+$$

$$+2\,({}^t(t_0+t')St)))\cdot\vartheta^{(1)}\,(Z,S\,|\,t')\,|_k\,\breve{\omega}_1^{(5)})=$$
$$=\exp(\pi iq^{-2}S[t_0])(\det S)^{-1}i^{-r}\Phi(\sum_{t,t'\in T_{1,q}(S)}\exp(\pi iq^{-2}(S[t]bN+$$
$$+S[t'](\gamma-(ap)^{N-1})+2\,({}^t(t_0+t')St)))\vartheta^{(1)}\,(Z,S\,|\,t')\,)=$$
$$=\exp(\pi iq^{-2}S[t_0])(\det S)^{-1}i^{-r}\sum_{t\in T_{1,q}(S)}\exp(\pi iq^{-2}(S[t]bN+2\,{}^tt_0St)).$$

If we consider the "change of variables" $t=\tilde{t}+pt_0$ we see that

$$\Phi(\vartheta^{(1)}\,(Z,S\,|\,t_0)\,|_k\,\breve{\omega}_1)=$$
$$=\exp(\pi iq^{-2}S[t_0])(\det S)^{-1}i^{-r}\sum_{\tilde{t}\in T_{1,q}(S)}\exp(\pi iq^{-2}((S[\tilde{t}]+$$
$$+p^2S[t_0]+2p\,{}^tt_0St)bN+2\,{}^tt_0S\tilde{t}+2pS[t_0]))=$$
$$=\exp(\pi iq^{-2}(S[t_0]+2pS[t_0]))(\det S)^{-1}i^{-r}\cdot$$
$$\sum_{t\in T_{1,q}(S)}\exp(\pi iq^{-2}(S[t]bN+2\,{}^tt_0St)).$$

Hence, the previous expression vanishes unless t_0 satisfies the condition: $S[t_0]\equiv 0 \bmod qN$. For such a t_0 we have moreover:

$$\exp(\pi iq^{-2}S[t_0])\sum_{t\in T_{1,q}(S)}\exp(\pi iq^{-2}(S[t]bN+2\,{}^tt_0St))=$$
$$=\exp(\pi iq^{-2}S[t_0]ap)\sum_{t\in T_{1,q}(S)}\exp(\pi iq^{-2}(S[t+t_0]bN+2\,{}^tt_0Stap))=$$
$$=\sum_{t\in T_{1,q}(S)}\exp(\pi iq^{-2}(S[t]bN+2\,{}^tt_0Stap))=$$
$$=\sum_{t=Nt_1+pt_2}\exp(\pi iq^{-2}(S[t]bN+2\,{}^tt_0Stap))=$$

(where $Nt_1\in\frac{qS^{-1}\mathbb{Z}^{(r)}\cap N\mathbb{Z}^{(r)}}{q\mathbb{Z}^{(r)}}$ and $pt_2\in\frac{qS^{-1}\mathbb{Z}^{(r)}\cap p\mathbb{Z}^{(r)}}{q\mathbb{Z}^{(r)}}$)

$$=\sum_{Nt_1+pt_2}\exp(\pi iq^{-2}(S[Nt_1]bN+2p\,{}^tt_{0,2}Spt_2ap))$$

(where $t_0=Nt_{0,1}+pt_{0,2}$).

We have thus to understand for which values of $t_{0,2}$ the function of the variable t_2: $\exp(2ap\pi iq^{-2}p\,{}^tt_{0,2}Spt_2)$, is the trivial element in the group $\left(\frac{qS^{-1}\mathbb{Z}^{(r)}\cap p\mathbb{Z}^{(r)}}{q\mathbb{Z}^{(r)}}\right)^*$. The multiplication by S/q induces an isomorphism between $T_{1,q}(S)$ and $\frac{\mathbb{Z}^{(r)}}{S\mathbb{Z}^{(r)}}$. The image of $\frac{qS^{-1}\mathbb{Z}^{(r)}\cap p\mathbb{Z}^{(r)}}{q\mathbb{Z}^{(r)}}$ through this isomorphism is of course $\frac{\mathbb{Z}^{(r)}\cap\frac{S}{N}\mathbb{Z}^{(r)}}{S\mathbb{Z}^{(r)}}$. Let K be an integral matrix in $GL(r,\mathbb{Q})$ such that $\mathbb{Z}^{(r)}\cap\frac{S}{N}\mathbb{Z}^{(r)}=K\mathbb{Z}^{(r)}$; since qS^{-1} is integral, $K\mathbb{Z}^{(r)}\supset p\mathbb{Z}^{(r)}$. Hence if $\exp(2\pi i\frac{ap^2}{N}\,{}^tt_{0,2}k)$ defines the trivial element of $\left(\frac{K\mathbb{Z}^{(r)}}{S\mathbb{Z}^{(r)}}\right)^*$

then $\frac{ap^2}{N} t_{0,2} \in p^{-1}\mathbb{Z}^{(r)}$ and in particular $t_{0,2} \equiv 0 \bmod N$. Viceversa this condition implies the triviality of the character.

As a result of our analysis we have that

$$\Phi(\vartheta^{(1)}(Z,S\,|\,t_0)\;|_k\,\breve{\omega}_1) = \begin{cases} \Phi(\vartheta^{(1)}(Z,S\,|\,0)\;|_k\,\breve{\omega}_1) & \text{if } t_0 \equiv 0 \bmod N; \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand we know from [**B.F.S.-P.**] that

$$\Phi(\vartheta^{(1)}(Z,S\,|\,0)\;|_k\,\breve{\omega}_1) = s_p(\mathbb{Q}^r)\det\nolimits_p(S)^{-\frac{1}{2}}.$$

(We are assuming $\det_p(S) = p^{2l}$). ∎

4.7. Remark It is possible to check by somewhat tedious calculations that the result in the previous lemma can be extended to every integer (not only odd integers) $q = p^k N$ where as usual p is a prime, $(N,p) = 1$ and $k \geq 1$.

The statement in the previous lemma is exactly what we need to conclude the analysis in the odd q case. In fact the action of the matrices of the quotient $\frac{\Gamma_{n+1}[N]}{\Gamma_{n+1}[q]} \cong Sp(n+1,\mathbb{F}_p)$ just affect the columns of the lifted characteristic $\bmod\, p$, hence we can go along the given proof for $q = p$, and obtain an analogous result (see the final Theorem 4.10).

4.8. Remark With minor changes the previous argument works also for even values of q (but for the moment we still consider only odd primes). Some difficulties arise from the fact that in this case we have to consider quadratic forms S that are integral and not even. For this reason some theta transformation formulas could be no longer valid. As to the translations, $T(B) = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \in \Gamma[N]$, it is easy to check that, since $N \equiv 0 \bmod 2$, we can still use the habitual formulas. A little bit more careful we have to be when we deal with the 1-involution $\breve{\omega}_1$. In order to go around this obstacle we observe that we can express our theta series as follows,

$$\vartheta_P(Z,S\,|\,T) = \sum_{L \bmod 2} \vartheta_P(Z,4S\,|\,2T + 2qL)\,. \tag{17}$$

So, applying Lemma 4.6 for

$$\breve{\omega}_1 \equiv \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \bmod p; \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \bmod 4N, \end{cases}$$

(see also Remark 4.7) yields:

$$\Phi(\vartheta_{P_1}(Z,S\,|\,t_0 T)\ |\breve{\omega}_1^{\uparrow}) =$$
$$= \Phi(\sum_{l \bmod 2} \vartheta(Z,4S\,|\,2t_0+2ql)\ |\breve{\omega}_1)\vartheta_P(Z,S\,|\,T) =$$
$$= \begin{cases} s_p(\mathbb{Q}^r)\det_p(S)^{-\frac{1}{2}}\vartheta_P(Z,S\,|\,T) & \text{if } t_0 \equiv 0 \bmod N; \\ 0 & \text{otherwise.} \end{cases}$$

Indeed $\det_p(4S) = \det_p(S)$ and there exists one $l \bmod 2$ such that $2ql + 2t_0 \equiv 0 \bmod 4N$ if and only if $t_0 \equiv 0 \bmod N$.
In this way we see that it is possible to extend the given proof to the case of even levels and an odd prime.

4.9. Remark The worst situation occurs, as usual, when we have to deal with the prime 2. Let us see how we can provide an adapted version of our proof for this last case.
We shall consider integral quadratic forms $S = S^r$ such that qS^{-1} is even, where $q = 2\cdot N$ and $N \equiv 1 \bmod 2$. Moreover we can assume that S is not even (because this is the only problematic case).

We now go on in the same way as in Remark 4.8 considering expressions for the theta series $\vartheta_P(Z,S\,|\,T)$ like in (17).
It is then easy to check that the condition $S[T] \equiv 0 \bmod 2q$ ensures that the trace restricted to the subgroup of the translations is not zero. Then, we would like to establish a commutation formula between the trace and the Siegel operators on the lifted theta series $\vartheta_{P_1}(Z,S\,|\,0\,T)$. Of course we shall require that the characters satisfy the condition: $v_S|_{\Gamma[q]} = v_Q|_{\Gamma[q]} = 1$; we must point out that in particular this implies that $\det_2(S) = 2^{2l}$ for some non-negative integer l.
If we follow again the calculations we have made in the odd prime case we can see that we meet no further difficulties till when we have to evaluate the contributions (relevant to the "$\bar{P}^1\omega_1$ sum") of this kind:

$$\Phi(\sum_{\beta_1,\beta_2} \vartheta_{P_1}^{(n+1)}(Z,S\,|\,[0,T]m_1(x_{k+1})|_{k,\rho_1}T(\begin{pmatrix} \beta_1 & \beta_2 \\ {}^t\beta_2 & 0 \end{pmatrix})\omega_1).$$

By using expansion (17), we can express the sum above as follows

$$\cdots = \Phi(\sum_{\beta_1,\beta_2}\ \sum_{H=(h_1,\cdots,h_{k+1},H_{k+2}) \bmod 2} \vartheta_{P_1}^{(n+1)}(Z,4S\,|\,[2qh_1+2t_k,\cdots$$
$$\cdots, 2qH_{k+2}+2t_kx_{k+1}+2T_{k+1}])|_{k,\rho_1}T(\begin{pmatrix} \beta_1 & \beta_2 \\ {}^t\beta_2 & 0 \end{pmatrix})\omega_1) =$$
$$= \Phi(2^n\sum_{\beta_1}\sum_{h_1}\exp(\pi i(S[h_1]+S[\tfrac{t_k}{q}])\beta_1)\vartheta_{P_1}^{(n+1)}(Z,4S\,|\,[2qh_1+2t_k,\cdots$$
$$\cdots, 2qH_{k+2}+2t_kx_{k+1}+2T_{k+1}])|_{k,\rho_1}\omega_1) =$$

$$= 2^{n+1}2^{r-1}s_2(\mathbb{Q}^r)\det{}_2(4S)^{-\frac{1}{2}}\vartheta_P^{(n)}(Z,S\,|\,[t_1,\cdots,t_{k-1},0,t_kx_{k+1}+T_{k+1}]) =$$

$$= 2^n s_2(\mathbb{Q}^r)\det{}_2(S)^{-\frac{1}{2}}\vartheta_P^{(n)}(Z,S\,|\,[t_1,\cdots,t_{k-1},0,t_kx_{k+1}+T_{k+1}]) =.$$

We realize therefore that in comparison with the other cases we get in every such addend an extra-factor $\frac{1}{2}$. In conclusion we achieve a slightly different commutation formula, i.e.:

$$\Phi(Tr^{(n+1)}\vartheta_{P_1}(Z,S\,|\,[0,T])) = \tag{18}$$

$$= 2^{2n+1}Tr^{(n)}(((2^{n-\iota+1}-1)+\det{}_2(S)^{-\frac{1}{2}}s_2(\mathbb{Q}^r)2^n(2^{n+1}-2^\iota))$$
$$\vartheta_P(Z,S\,|\,T)) + \sum_j 2^{3n+1}\det{}_2(S)^{-\frac{1}{2}}s_2(\mathbb{Q}^r)Tr^{(n)}(\vartheta_P(Z,S\,|\,\mathcal{T}_j))$$

(where, as above, $\mathcal{T}_j$ are certain characteristics whose rank $\bmod p$ is less than the rank $\bmod p$ of T).

We would like to summarize the content of this last subsection in the following final statement.

4.10. Theorem *Let $r=2k$, k a positive integer, and let ρ_0 be a rational reduced representation of $GL(n,\mathbb{C})$, $\rho=[\rho_0,r]$. Fix the integer $q=N\cdot p$, with p prime and $(N,p)=1$. Call $\Theta\,[\Gamma_n\,[N]]_{\rho,\chi_N}\subset[\Gamma_n\,[N]\,,\rho,\chi_N]$ the vector space spanned by theta series of the type $\vartheta_P^{(n)}\,(Z,Q\,|\,T)$; assume that χ_N satisfies Freitag's conditions (see* [**Fr.1**]*). Consider moreover $\vartheta_P^{(n)}\,(Z,S\,|\,T)\in[\Gamma_n\,[q]\,,\rho,v_S]$ such that $v_S|_{\Gamma_0^0[q]\cap\Gamma[N]}=\chi_N|_{\Gamma_0^0[q]\cap\Gamma[N]}$.*

If $S[T]$ is not 0 modulo qp, then the trace vanishes.

If we denote by ι the rank of $T \bmod p$ and define ϵ to be -1 for $p=2$ and S not even, and 0 otherwise, the following condition:

$$1+s_p(\mathbb{Q}^r)\det{}_p(S)^{-\frac{1}{2}}p^{\xi+1+\epsilon}\neq 0\ \forall n\leq\xi\leq r+\iota,$$

implies that:

$$Tr^{\Gamma_n[q],1}_{\Gamma_n[N],1}\vartheta_P^{(n)}\,(Z,S\,|\,T)\in\Theta\,[\Gamma_n\,[N]]_\rho\,.$$

It has been pointed out in [**B.F.S.-P.**] that, if the ambient quadratic space contains even lattices of level q, the condition $s_p(\mathbb{Q}^r)=1$ is equivalent to the fact that $\mathbb{Q}^r$ carries even lattices of level N. Therefore, at least for q odd, if $s_p(\mathbb{Q}^r)\neq 1$ the fulfillment of the condition which appears in the previous statement implies the vanishing of the trace.

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