LINEAR RELATIONS BETWEEN MODULAR FORM COEFFICIENTS AND NON-ORDINARY PRIMES

YOUNGJU CHOIE, WINFRIED KOHNEN, AND KEN ONO

Abstract. Here we generalize a classical observation of Siegel by determining all the linear relations among the initial Fourier coefficients of a modular form on $SL_2(\mathbb{Z})$. As a consequence, we identify spaces M_k in which there are universal p-divisibility properties for certain p-power coefficients. As a corollary, let

$$f(z) = \sum_{n=1}^{\infty} a_f(n) q^n \in S_k \cap O_L[[q]]$$

(note: $q := e^{2\pi i z}$) be a normalized Hecke eigenform, and let $k \equiv \delta(k) \pmod{12}$, where $\delta(k) \in \{4, 6, 8, 10, 14\}$. Reproducing earlier results of Hatada and Hida, if p is a prime for which $k \equiv \delta(k) \pmod{p-1}$, and $\mathfrak{p} \subset O_L$ is a prime ideal above p, then we show that

$$a_f(p) \equiv 0 \pmod{\mathfrak{p}}.$$

1. Introduction and statement of results

If $k \geq 4$ is even, then let M_k (resp. S_k) denote the finite dimensional $\mathbb{C}$-vector space of weight k holomorphic modular forms (resp. cusp forms) on $SL_2(\mathbb{Z})$ (see [7] for background on modular forms). As usual, we identify a modular form $f(z)$ by its Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_f(n) q^n,$$

where $q := e^{2\pi i z}$. As is customary, let $\Delta(z) \in S_{12}$ be the cusp form

$$\Delta(z) := q \prod_{n=1}^{\infty} (1 - q^n)^{24} = q - 24q^2 + \cdots, \tag{1.1}$$

and for even $k \geq 4$ let $E_k(z) \in M_k$ be the normalized Eisenstein series

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \left(\sum_{1 \leq d \mid n} d^{k-1} \right) q^n. \tag{1.2}$$

Here B_k is the k-th Bernoulli number. For convenience, we let $E_0(z) := 1$. Throughout, if $k \geq 4$ is even, then let (for example, see I.2 of [7])

$$d(k) := \dim_{\mathbb{C}}(M_k) = \begin{cases} \lfloor k/12 \rfloor + 1 & \text{if } k \not\equiv 2 \pmod{12}, \\ \lfloor k/12 \rfloor & \text{if } k \equiv 2 \pmod{12}. \end{cases} \tag{1.3}$$

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Furthermore, define $\delta(k) \in \{0, 4, 6, 8, 10, 14\}$ by the congruence

$$\delta(k) \equiv k \pmod{12}. \tag{1.4}$$

If N is a non-negative integer, then we let

$$L_{k,N} := \{(c_0, c_1, \ldots, c_{d(k)+N}) \in \mathbb{C}^{d(k)+N+1} \ : \ \sum_{\nu=0}^{d(k)+N} c_\nu a_f(\nu) = 0 \quad \forall\, f(z) = \sum_{n=0}^{\infty} a_f(n) q^n \in M_k\} \tag{1.5}$$

be the space of linear relations satisfied by the first $d(k)+N+1$ Fourier coefficients of all the forms $f(z) \in M_k$. In his study of Hilbert modular forms, Siegel [8] determined the spaces $L_{k,0}$. To state our results, for each $g(z) \in M_{12N}$, define numbers $b(k, N, g; \nu)$ by

$$\frac{E_{14-\delta(k)}(z)}{\Delta(z)^{d(k)+N}} \cdot g(z) = \sum_{\nu=0}^{d(k)+N} b(k, N, g; \nu) q^{-\nu} \ + \ \sum_{\nu=1}^{\infty} c(k, N, g; \nu) q^{\nu}. \tag{1.6}$$

The numbers $b(k, N, g; \nu)$ are the Fourier coefficients of the "principal part", together with the constant term, of the modular form above. In this notation, we have the following characterization of the $L_{k,N}$.

Theorem 1.1. *The map $\phi_{k,N} : M_{12N} \to L_{k,N}$ defined by*

$$\phi_{k,N}(g(z)) = \{b(k, N, g; \nu) \ : \ \nu = 0, \ldots, d(k) + N\}$$

defines a linear isomorphism between M_{12N} and $L_{k,N}$.

As a corollary to Theorem 1.1, we consider the distribution of non-ordinary primes for normalized Hecke eigenforms. First we recall the following well known problem (see Gouvêa's expository article [2]).

Problem. Suppose that $f(z) = \sum_{n=1}^{\infty} a_f(n) q^n \in S_k$ is a normalized Hecke eigenform. A prime p is *non-ordinary* for $f(z)$ if $a_f(p) \equiv 0 \pmod{p}$. Are there infinitely many non-ordinary primes for $f(z)$?

Although there are strong results on the more general problem for very special modular forms on congruence subgroups $\Gamma_0(N)$ (e.g. such as CM cusp forms, and weight 2 newforms associated to elliptic curves over $\mathbb{Q}$), little is known. Using Theorem 1.1, we obtain elementary results related to this question. The following theorem applies for all forms when $p = 2$ and 3, and requires that $\delta(k) \neq 0$ for primes $p \geq 5$.

Theorem 1.2. *Let p be prime, and suppose that $f(z) = \sum_{n=0}^{\infty} a_f(n) q^n \in M_k \cap O_L[[q]]$, where O_L denotes the algebraic integers of a number field L.*

(1) *If $p \in \{2, 3\}$, and $b \geq 1$ is an integer for which $12p^b - 2 \geq k$, then*

$$a_f(p^b) \equiv 0 \pmod{p}.$$

(2) *Suppose that $p \geq 5$, and that $\delta(k) \in \{4, 6, 8, 10, 14\}$. If $b \geq 1$ is an odd integer, and $a \geq 0$ is an integer for which*

$$k = (\delta(k) - 2)p^b + 2 - a(p - 1),$$

then

$$a_f(p^b) \equiv -(24 + \alpha_k) a_f(0) \pmod{p},$$

where

$$\alpha_k := \frac{-2(14 - \delta(k))}{B_{14-\delta(k)}} \in \mathbb{Z}.$$

Remark. Theorem 1.2 does not include cases where $p \geq 5$ is prime and $\delta(k) = 0$. The condition on k in the statement of Theorem 1.2 (2) never holds when $\delta(k) = 0$. More to the point, the conclusion of Theorem 1.2 (2) does not always hold. For example, $p = 13$ is an ordinary prime for $\Delta(z)$.

Theorem 1.2 allows us to reproduce some results of Hatada [3] (in the case where $p = 2$ and 3) and Hida [4, 5, 6] (for primes $p \geq 5$) on non-ordinary primes.

Corollary 1.3. *Let p be prime, and suppose that $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_k$ is a normalized Hecke eigenform. Let L_f be the number field generated by the coefficients of $f(z)$, and let $\mathfrak{p} \subset O_{L_f}$ be any prime ideal above p.*

(1) *If $p = 2$ or 3, then*

$$a_f(p) \equiv 0 \pmod{\mathfrak{p}}.$$

(2) *If $p \geq 5$ is prime, $\delta(k) \in \{4, 6, 8, 10, 14\}$ and $k \equiv \delta(k) \pmod{p-1}$, then*

$$a_f(p) \equiv 0 \pmod{\mathfrak{p}}.$$

The primes underlined in the table (see page 203 of [2]) below are examples of Corollary 1.3.

Eigenform	Primes $p \leq 10^6$ for which $a_f(p) \equiv 0 \pmod{p}$
$\Delta(z)$	$\underline{2}$, $\underline{3}$, 5, 7, 2411
$\Delta(z)E_4(z)$	$\underline{2}$, $\underline{3}$, $\underline{5}$, $\underline{7}$, 11, $\underline{13}$, 59, 15271, 187441
$\Delta(z)E_6(z)$	$\underline{2}$, $\underline{3}$, $\underline{5}$, $\underline{7}$, 11, $\underline{13}$
$\Delta(z)E_8(z)$	$\underline{2}$, $\underline{3}$, $\underline{5}$, $\underline{7}$, 11, $\underline{13}$, 17, 3371, 64709
$\Delta(z)E_{10}(z)$	$\underline{2}$, $\underline{3}$, $\underline{5}$, $\underline{7}$, $\underline{13}$, 17, 19
$\Delta(z)E_{14}(z)$	$\underline{2}$, $\underline{3}$, $\underline{5}$, $\underline{7}$, 11, $\underline{13}$, 17, 19, 23

2. Proof of Theorem 1.1

The proof is a generalization of a method of Siegel [8] where the $N = 0$ case is treated.

Proof of Theorem 1.1. Let us first show that

$$\sum_{\nu=0}^{d(k)+N} b(k, N, g; \nu) a_f(\nu) = 0 \tag{2.1}$$

for all $g \in M_{12N}$ and all $f(z) = \sum_{\nu=0}^{\infty} a_f(\nu) q^\nu \in M_k$. If we let

$$G(z) := \frac{E_{14-\delta(k)}}{\Delta(z)^{d(k)+N}} \cdot g(z),$$

then (2.1) is equivalent to the assertion that the constant term of the series Gf is zero. The dimension formula (1.3) implies that

$$\frac{fg}{E_{\delta(k)}\Delta^{d(k)+N-1}}$$

is a meromorphic modular function on $SL_2(\mathbb{Z})$ of weight zero. Since $k \equiv \delta(k) \pmod{12}$, we find from the valence formula (for example, see page 6 of [7]) that $f/E_{\delta(k)}$ is holomorphic on $\mathcal{H}$. Therefore it follows that

$$\frac{fg}{E_{\delta(k)}\Delta^{d(k)+N-1}}$$

is a polynomial in the Hauptmodule $j(z)$:

$$j(z) = \frac{E_4(z)^3}{\Delta(z)} = q^{-1} + 744 + \cdots .$$

On the other hand, an easy calculation (for example, see equation (8) of [8]) reveals that

$$-\frac{1}{2\pi i}\frac{d}{dz}j = \frac{E_{14}}{\Delta}.$$

Moreover, we have that

$$j^m \frac{d}{dz}j = \frac{1}{m+1}\frac{d}{dz}j^{m+1} \qquad (m \in \mathbb{Z},\ m \geq 0)$$

has constant term zero. Therefore

$$Gf = \frac{fg}{E_{\delta(k)}\Delta^{d(k)+N-1}} \cdot \frac{E_{14}}{\Delta}$$

has constant term zero, and this confirms (2.1).

Suppose that $f(z) = \sum_{n=0}^{\infty} a_f(n)q^n \in M_k$. The map $\phi_{k,N}$ is clearly linear, and it is injective, since a modular form of weight $\ell \leq 2$ (and vanishing at infinity if $\ell = 0$) is identically zero. To complete the proof, recall the well-known fact (see part I, chap. III, sect. 4 of [7]) that the $d(k)$ functionals $\{a_f(0), a_f(1), \ldots, a_f(d(k)-1)\}$ form a basis of the dual space M_k^*. Therefore, it follows that

$$\dim L_{k,N} = N + 1 = \dim M_{12N}.$$

This proves the theorem. □

Remark. If $k \equiv 2 \pmod 4$, then $E_{14-\delta(k)}$ is either 1, E_4 or E_8 and hence has positive Fourier coefficients. Therefore, taking $N = 0$, we obtain a linear relation

$$\sum_{\nu=0}^{d(k)} c_\nu a_f(\nu) = 0$$

between the first $d(k)+1$ Fourier coefficients of modular forms in M_k where all the c_ν are strictly positive (this was observed in [8]). In particular, this implies that for $k \equiv 2 \pmod 4$ the first sign change of the Fourier coefficients of a non-zero cusp form $f \in M_k$ with real Fourier coefficients already occurs among the first $d(k)+1$ coefficients (and this bound is sharp, too, as is easily seen).

If $k \equiv 0 \pmod 4$, the above reasoning breaks down. To our knowledge, an answer to the corresponding question on the first sign change remains open in these cases[1].

Remark. A similar result as stated in the Theorem 1.1 can certainly be proved for modular forms on genus zero subgroups of $SL_2(\mathbb{Z})$ (and in particular for half-integral weight modular forms of level 4).

[1]For some general results about sign changes of cusp forms on rather general subgroups of $SL_2(\mathbb{R})$, we refer to [1].

3. Proof of Theorem 1.2 and Corollary 1.3

We begin by restating one of the main conclusions of Theorem 1.1 in a convenient form.

Theorem 3.1. *If $f(z) \in M_k$ and $g(z) \in M_{12N}$, then the constant term of*

$$\frac{E_{14-\delta(k)}(z)g(z)}{\Delta(z)^{d(k)+N}} \cdot f(z)$$

is zero.

Proof of Theorem 1.2. First we prove Theorem 1.2 (2). Define $g(z)$ by

$$g(z) := E_{14-\delta(k)}(z)^{p^b-1} \cdot E_{p-1}(z)^a. \tag{3.1}$$

The fact that $b \geq 1$ is odd implies that $g(z) \in M_{12N}$, where

$$N = \frac{(p^b-1)(14-\delta(k)) + a(p-1)}{12} \in \mathbb{Z}_+. \tag{3.2}$$

To see this, observe that the given representation of k implies that

$$a(p-1) = -(k-2) + (\delta(k)-2)p^b \equiv (p-1)(k-2) \pmod{12}.$$

Formula (3.2) combined with (1.3) and the given representation of k implies that

$$d(k) + N = p^b.$$

Theorem 3.1, combined with the fact (see page 164 of [7]) that

$$E_{p-1}(z) \equiv 1 \pmod{p}$$

shows that the constant term of

$$\begin{aligned}\frac{E_{14-\delta(k)}(z)^{p^b} E_{p-1}(z)^a}{\Delta(z)^{p^b}} \cdot f(z) &\equiv \frac{E_{14-\delta(k)}(p^b z)}{\Delta(p^b z)} \cdot f(z)\\ &\equiv \left(q^{-p^b} + 24 + 324q^{p^b} + \cdots\right)\left(1 + \alpha_k q^{p^b} + \cdots\right) \cdot f(z)\\ &\equiv \left(q^{-p^b} + (24+\alpha_k) + \cdots\right) \cdot \left(\sum_{n=0}^{\infty} a_f(n) q^n\right) \pmod{p}\end{aligned}$$

is zero modulo p. The conclusion of Theorem 1.2 (2) follows immediately.

To prove Theorem 1.2 (1), one argues as in the $\delta(k) = 14$ and $N = 0$ case above. In this case, we have

$$E_{14-\delta(k)}(z) = E_0(z) = 1.$$

One simply replaces $E_{p-1}(z)^a$ by

$$E_{12p^b+2-k}(z) \in M_{12p^b+2-k}$$

in (3.1). Here we require that $12p^b + 2 - k \geq 4$. The congruence for $E_{p-1}(z)$ is replaced by the universal congruence

$$E_k(z) \equiv 1 \pmod{24}.$$

□

Proof of Corollary 1.3. Suppose that $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_k$ is a normalized Hecke eigenform. We begin by establishing, in each case, that there is a positive integer b for which

$$a_f(p^b) \equiv 0 \pmod{p}. \tag{3.3}$$

If $p = 2$ or 3, then Theorem 1.2 (1) implies (3.3) since $a_f(0) = 0$. If $p \geq 5$ is prime, then there are integers $1 \leq b \equiv 1 \pmod{2}$ and $a \geq 0$ for which

$$k = (\delta(k)-2)p^b + 2 - a(p-1) = (\delta(k)-2)(p-1+1)^b + 2 - a(p-1) \equiv \delta(k) \pmod{p-1}.$$

By Theorem 1.2 (2), since $a_f(0) = 0$, we obtain (3.3).

The definition of the Hecke operators imply, for every non-negative integer n, that

$$a_f(p)a_f(p^n) = a_f(p^{n+1}) + p^{k-1}a_f(p^{n-1}) \equiv a_f(p^{n+1}) \pmod{p}.$$

By induction, we have that

$$a_f(p^b) \equiv a_f(p)^b \pmod{p}.$$

Corollary 1.3 follows immediately from the truth of (3.3). □

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Department of Mathematics, Pohang Institute of Science and Technology, Pohang 790-784 Korea

E-mail address: yjc@postech.ac.kr

Universität Heidelberg, Mathematisches Institut, INF 288, D-69120, Heidelberg, Germany

E-mail address: winfried@mathi.uni-heidelberg.de

Department of Mathematics, University of Wisconsin, Madison, WI 53706

E-mail address: ono@math.wisc.edu